ON THE COMBINED EFFECT OF A WEDGE AND STAMP ON AN ELASTIC HALF-PLANE

(O SOVMESTNOM DEISTVII NA UPRUGUIU POLUPLOSKOST' KLINA I SHTAMPA)

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1. **Fundamental dependences.** Let us start from the following easily verified representation of the solution of the equilibrium equations of the plane theory of elasticity in displacements:

$$2\mu u = \mathrm{Re}\,[\varkappa_0\Phi + iy\,\Phi' + (1+\varkappa_0)\,\Psi - x\Psi']$$
$$2\mu v = -\,\mathrm{Re}\,[(1+\varkappa_0)\,\Phi - iy\,\Phi' + \varkappa_0\Psi + x\Psi']\,i \qquad \left(\varkappa_0 = \frac{\mu}{\lambda+\mu}\right) \qquad (1.1)$$

Here Φ, Ψ are arbitrary analytic functions of the complex variable $z = x + iy$, defined in the domain of a section of the body or plate. The solution (1.1) may be found by superposition of the first and second solutions and discarding excess functions [1]. We obtain

$$\sigma_x = \mathrm{Re}\,[\Phi' + iy\Phi'' + \Psi' - x\Psi''],$$
$$\sigma_y = \mathrm{Re}\,[\Phi' - iy\Phi'' + \Psi' + x\Psi''] \qquad (1.2)$$
$$\tau_{xy} = -\,\mathrm{Re}\,[y\Phi'' + ix\Psi'']$$

The relationships

$$2\mu\frac{\partial v}{\partial x} = -(1+\varkappa_0)\,\mathrm{Re}\,i\,(\Phi' + \Psi') + \tau_{xy}$$
$$2\mu\frac{\partial u}{\partial y} = (1+\varkappa_0)\;\;\mathrm{Re}\;\;i\,(\Phi' + \Psi') + \tau_{xy} \qquad (1.3)$$

hold.

2. **Representation of the solution for a half-plane.** Let a smooth, thin, symmetric, absolutely stiff wedge of given shape be inserted on the portion L_x along the x-axis in an elastic half-plane $x \geq 0$. Only a normal loading of intensity $-p(y)$ which is symmetric relative to the origin is applied on the half-plane boundary $x = 0$. Because of the symmetry of the loading, the tangential stresses equal zero on the x-axis. Consequently, the elastic displacements of points in the direction of the y-axis equal zero on the x-axis outside the wedge.

The absence of tangential stresses at points of the x- and y-axes leads to the conditions (2.1)

$$\mathrm{Re}\, i\Phi' = 0, \qquad x=0, -\infty<y<\infty, \qquad \mathrm{Re}\, i\Psi' = 0, \qquad y=0, \quad 0<x<\infty$$

Hence, it also follows from (1.3) that the function $\Phi'(z)$ is continued analytically through the y-axis as well as through those portions of the x-axis where $\partial v/\partial x$ vanishes. Hence, the imaginary parts of Φ' take values of opposite sign at points of the x-axis symmetric to the origin. On the basis of (2.1) and (1.3) we arrive at the problem: in the upper half-plane to find an analytic function $\Phi'(z)$, which vanishes at infinity, according to the following condition on the two portions L_x of the x-axis, which are symmetrically disposed relative to the origin:

$$\mathrm{Re}\, i\Phi' = -q_0 v_x', \qquad q_0 = \frac{2\mu}{1+\varkappa_0} \tag{2.2}$$

We write its solution as

$$\Phi'(z) = \frac{2q_0}{\pi}\int_{L_x} \frac{x v_x'\, dx}{x^2 - z^2} \tag{2.3}$$

Here L_x is the portion lying to the right of the origin. Conditions on the y-axis

$$\mathrm{Re}\,\Psi'(z) = \sigma_x^\circ(y), \qquad \sigma_x^\circ(-y) = \sigma_x^\circ(y) \tag{2.4}$$

follow from the properties of the function $\Psi'(z)$.

Hence

$$\Psi'(z) = \frac{2z}{\pi}\int_0^\infty \frac{\sigma_x^\circ\, dy}{y^2+z^2} \tag{2.5}$$

On the y-axis we have

$$\sigma_x = -p(y) \tag{2.6}$$

Satisfying this boundary condition, we obtain

$$\frac{2q_0}{\pi}\frac{d}{dy}\left(y\int_{L_x}\frac{x v_x'\, dx}{x^2+y^2}\right) + \sigma_x^\circ = -p(y) \tag{2.7}$$

Substituting the value of σ_x° into (2.5), we find after having taken an intermediate integral

$$\Psi'(z) = -\frac{2q_0}{\pi}\int_{L_x}\frac{x v_x'\, dx}{(x+z)^2} - \frac{2z}{\pi}\int_0^\infty \frac{p(y)\, dy}{y^2+z^2} \tag{2.8}$$

For specified v_x' and $p(y)$, Formulas (2.3) and (2.8) yield the solution of the problem of a stiff wedge and an additional loading acting on a half-plane. For $p(y) = 0$, in particular, we obtain the solution of the problem of cleavage of a half-plane without taking account of crack formation. This solution does not demand knowledge of the second derivative [1] of $v(x, 0)$. Let us consider the following examples.

1. Parabolic wedge driven in along the x-axis to a depth H. On the wedge portion we have

$$v_x' = -\frac{h}{2\sqrt{H}\,\sqrt{H-x}} \tag{2.9}$$

The derivative is $v_x' = 0$ outside this portion on the x-axis. We obtain

$$\Phi'(z) = -\frac{hq_0}{\pi\sqrt{H}}\int_0^H \frac{x\,dx}{\sqrt{H-x}\,(x^2-z^2)}, \quad \Psi'(z) = \frac{hq_0}{\pi\sqrt{H}}\int_0^H \frac{x\,dx}{\sqrt{H-x}\,(x+z)^2} \qquad (2.10)$$

or

$$\Phi'(z) = -\frac{q_0 h}{2\pi H}[\chi(\xi_1)-\chi(\xi_2)], \qquad \Psi'(z) = \frac{q_0 h}{\pi(H+z)}[\chi(\xi_1)-1] \qquad (2.11)$$

$$\chi(\xi) = \xi\ln\frac{1-\xi}{1+\xi}, \quad \xi_1^2 = \frac{H}{H+z}, \quad \xi_2^2 = \frac{H-z}{H}$$

2. A half-plane reinforced along the x-axis by a system of m stiff, thin insertions of elliptic shape with semi-axes h_j, l_j. We obtain

$$v_{xj}' = -\frac{h_j}{l_j}(x-H_j)[l_j^2-(x-H_j)^2]^{-1/2} \qquad (2.12)$$

where H_j is the distance from the center of the jth ellipse to the origin.

Substituting into (2.3) and (2.8) and evaluating the integrals, we find

$$\Phi(z) = q_0 i\sum_{j=1}^{m}\left[\sqrt{l_j^2-(z-H_j)^2} + \sqrt{l_j^2-(z+H_j)^2} + 2iz\right]$$

$$\Psi'(z) = q_0 z\sum_{j=1}^{m}\frac{h_j}{l_j}\left[\frac{i(z+H_j)}{\sqrt{l_j^2-(z+H_j)^2}}+1\right] \qquad (2.13)$$

If $l_j = h_j$ we then obtain the solution of the problem for a half-plane reinforced by thin circular insertions of different radii along the x-axis.

3. A half-plane reinforced along the x-axis by thin rectangular insertions. Here it is first necessary to consider insertions of constant thickness $2h_j^\circ$ with triangular tips of length l, and then, keeping the length of the insertion $2h_j$ unchanged, to pass to the limit permitting l to tend to zero. After evaluation we obtain

$$\Phi(z) = \frac{q_0}{\pi}\sum_{j=1}^{m} h_j^\circ \ln\frac{\tau_{1j}^2-1}{\tau_{2j}^2-1}, \qquad \tau_{1j} = \frac{z+h_j}{H_j}$$

$$\Psi(z) = \frac{4q_0}{\pi}z\sum_{j=1}^{m} h_j^\circ h_j[(H_j+z)^2-h_j^2]^{-1}, \qquad \tau_{2j} = \frac{z-h_j}{H_j} \qquad (2.14)$$

where H_j are the distances of the centers of the insertions from the origin.

3. Effect of a wedge and stamp on a half-plane. Let a thin smooth wedge of given shape which is symmetrical relative to the x-axis act on the L_x portion of the x-axis in the $x \geq 0$ half-plane, and a system of smooth stamps of given shape symmetrical relative to the origin, on the L_y portion of the y-axis.

According to (2.3) and (2.8)

$$\Phi'(z) = \frac{2q_0}{\pi}\int_{L_x}\frac{xv_x'\,dx}{x^2-z^2}, \quad \Psi'(z) = -\frac{2q_0}{\pi}\int_{L_x}\frac{xv_x'\,dx}{(x+z)^2} - \frac{1}{\pi i}\int_{L_y'}\frac{p(t)\,dt}{t-z} \qquad (3.1)$$

Here the second member in (2.8) has been rewritten in another form by using the variable $t = iy$; L_y' is traversed in the positive direction, and $p(t)$ is the pressure under the stamp.

On the y-axis we have

$$\tau_{xy} = 0, \qquad \operatorname{Re} i\Phi' = 0 \tag{3.2}$$

Hence, from the second relationship (1.3) we deduce the pressure on the section

$$\operatorname{Re} i\Psi' = q_0 u_y' \tag{3.3}$$

where u_y' is known. Therefore, substituting the value of Ψ' from (3.1), we find

$$-\frac{1}{\pi}\int_{L_y'} \frac{p(t)\,dt}{t-t_0} = q_0 u_y' + \frac{4q_0 y_0}{\pi}\int_{L_x} \frac{v_x' x^2\,dx}{(x^2+y_0^2)^2} \tag{3.4}$$

or, returning to the variable y

$$\frac{1}{\pi}\int_{L_y} \frac{p(y)\,dy}{y-y_0} = q_0 u_y' + \frac{4q_0 y_0}{\pi}\int_{L_x} \frac{x^2 v_x'\,dx}{(x^2+y_0^2)^2} \qquad (L_y = -L_y') \tag{3.5}$$

As an example, let us consider the simplest problem of a rectangular wedge and rectilinear stamp acting on an elastic half-plane.

If the wedge of thickness $2h$ penetrates a depth H, then by first writing the integral in the right side of (3.5) for a rectilinear wedge with a triangular tip of length l and then passing to the limit as l tends to zero, we may write taking into account that $u_y' = 0$

$$\int_{-b}^{b} \frac{p\,dy}{y-y_0} = -\frac{4hq_0H^2y_0}{(H^2+y_0^2)^2} \tag{3.6}$$

Here b is the half-width of the portion of the stamp in contact with the half-plane.

The most general solution of (3.6) is [2]

$$p(y_0) = \frac{4hq_0H^2}{\pi^2\sqrt{b^2-y_0^2}}\int_{-b}^{b} \frac{y\sqrt{b^2-y^2}\,dy}{(y^2+H^2)^2(y-y_0)} + \frac{P_0}{\pi\sqrt{b^2-y_0^2}} \tag{3.7}$$

Here P_0 is the force pressing the stamp to the half-plane. The integral in the first term can be evaluated. We obtain

$$p(y_0) = \frac{2q_0hH}{\pi\Delta\sqrt{b^2-y_0^2}}\,\frac{b^2H^2-(H^2+\Delta^2)y_0^2}{(y_0^2+H^2)^2} + \frac{P_0}{\pi\sqrt{b^2-y_0^2}} \qquad (\Delta = \sqrt{b^2+H^2}) \tag{3.8}$$

Formula (3.8) is valid if

$$P_0 > 2q_0hb^2H\,\Delta^{-3} \tag{3.9}$$

The edges of the stamp will hence come in contact with the half-plane.

If (3.9) is not satisfied, then $p(b) = 0$. Hence

$$P_0 = 2q_0hb^2H\,\Delta^{-3} \tag{3.10}$$

Substituting into (3.8) and computing, we obtain

$$p(y_0) = \frac{2q_0hH}{\pi\Delta^3}\,\frac{[H^2(H^2+\Delta^2)-b^2y_0^2]\sqrt{b^2-y_0^2}}{(y_0^2+H^2)^2} \tag{3.11}$$

For a specified value of P_0 the relationship (3.10) determines the size of the pressure section. The possibility of solving (3.11) for a rectilinear stamp is stipulated by the bulging of the half-plane boundary in the neighborhood of the origin under the influence of the penetrated wedge.

If a semi-infinite rectangular wedge is driven in to the right along the x-axis in the half-plane so that the distance between its tip and the boundary of the half-plane is H, then the plus sign must be taken in the right-hand side of (3.6) and we obtain for this case

$$p(y_0) = -\frac{2q_0hH}{\pi\Delta\sqrt{b^2-y_0^2}}\,\frac{b^2H^2-(H^2+\Delta^2)y_0^2}{(y_0^2+H^2)^2} + \frac{P_0}{\pi\sqrt{b^2-y_0^2}} \tag{3.12}$$

The pressure $p(y)$ turns out to be positive along the whole stamp under the condition

$$P_0 > 2q_0hb^2H^{-1}\Delta^{-1} \tag{3.13}$$

In the opposite case the pressure domain consists of the sections (a, b), $(-b, -a)$. Hence $p(\pm a) = 0$.

It is simplest to investigate this case by rewriting (3.6) as

$$\int_a^b \frac{p\,dy}{y^2 - y_0^2} = \frac{2hq_0H^2}{(H^2 + y_0^2)^2} \tag{3.14}$$

Substituting $y^2 = \xi$, $p_1(\xi) = p(\sqrt{\xi})\,/\,2\sqrt{\xi}$, we write

$$\int_{a^2}^{b^2} \frac{p_1(\xi)\,d\xi}{\xi - \xi_0} = \frac{2hq_0H^2}{(\xi_0 + H^2)^2} \tag{3.15}$$

Hence

$$p_1(\xi_0) = -\frac{hq_0H^2}{\pi^2}\left(\frac{\xi_0 - a^2}{b^2 - \xi_0}\right)^{1/2}\int_{a^2}^{b^2}\left(\frac{b^2 - \xi}{\xi - a^2}\right)^{1/2}\frac{d\xi}{(\xi^2 + H^2)^2(\xi - \xi_0)} \tag{3.16}$$

Evaluating the integral and returning to the old variable, we obtain

$$p(y_0) = \frac{2hq_0H^2y_0}{\pi(y_0^2 + H^2)^2}\left[1 + \frac{b^2 - a^2}{m_1\Delta^2}\left(\frac{1}{m_1 + 1} + \frac{y_0^2 + H^2}{2m_1^2\Delta^2}\right)\right]\left(\frac{y_0^2 - a^2}{b^2 - y_0^2}\right)^{1/2}$$

$$m_1^2 = \frac{a + H}{b + H} \tag{3.17}$$

Here under the radical we understand a branch which takes positive values on the upper edge of the slit (a, b).

The force pressing the stamp to the half-plane is

$$P_0 = \frac{2hq_0H^2(b^2 - a^2)}{(a^2 + H^2)^{3/2}\Delta} \tag{3.18}$$

If it has been given, then we find the size of the pressure section from the equality (3.18).

The existence of the solution (3.17) is here related to dropping of the half-plane boundary toward the wedge under the influence of the wedge. This drop, and in general, the change in the half-plane boundary under the influence of the wedge is characterized exactly by the second member in (3.5). If the pressing force is not large enough, a gap will remain between the domain boundary and the stamp.

It is easy to write down the solution of (3.5) in the general case for any u_y' and v_x'.

Let us note that for a given wedge there exists such a shape of a stamp (or stamps) for which the right-hand side of (3.5) vanishes. Namely, this will hold if

$$u_{y_0}' = -\frac{4y_0}{\pi}\int_{L_x}\frac{x^2v_x'\,dx}{(x^2 + y_0^2)^2} \tag{3.19}$$

In this case, for example, for one stamp we will obtain

$$p(y_0) = \frac{P_0}{\pi\sqrt{b^2 - y_0^2}} \tag{3.20}$$

where P_0 is the force pressing the stamp. The stamp boundary, hence, turns out to be concave if the right side of (3.19) is positive.

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